This is the March 2011 version of the High School Mathematics Model Curriculum for the conceptual category Algebra. (Note: The conceptual categories Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability do not directly align with high school mathematics courses. The current focus of this document is to provide instructional strategies and resources, and identify misconceptions and connections related to the clusters and standards. The Ohio Department of Education is working in collaboration with assessment consortia, national professional organizations and other multistate initiatives to develop common content elaborations and learning expectations.

Algebra	
Domain	**Cluster**
Seeing Structure in Expressions	Interpret the structure of expressions Write expressions in equivalent forms to solve problems.
Arithmetic with Polynomials and Rational Expressions	Perform arithmetic operations on polynomials. Understand the relationship between zeros and factors of polynomials. (+) Use polynomial identities to solve problems. Rewrite rational expressions
Creating Equations	Create equations that describe numbers or relationship.
Reasoning with Equations and Inequalities	Understand solving equations as a process of reasoning and explain the reasoning. Solve equations and inequalities in one variable. Solve systems of equations. Represent and solve equations and inequalities graphically.

High School Conceptual Category: Algebra

Domain	Seeing Structure in Expressions

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Cluster	***Interpret the structure of expressions***
Standards	1. Interpret expressions that represent a quantity in terms of its context. (a) Interpret parts of an expression, such as terms, factors, and coefficients. (b) Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1 + r)^n$ *as the product of P and a factor not depending on P.* 2. Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$*, thus recognizing it as a difference of squares that can be factored as* $(x^2 - y^2)(x^2 + y^2)$.

Instructional Strategies and Resources

Instructional Strategies

Extending beyond simplifying an expression, this cluster addresses interpretation of the components in an algebraic expression. A student should recognize that in the expression $2x + 1$, "2" is the coefficient, "2" and "x" are factors, and "1" is a constant, as well as "$2x$" and "1" being *terms* of the binomial expression**.** Development and proper use of mathematical language is an important building block for future content.

Using real-world context examples, the nature of algebraic expressions can be explored**.** For example, suppose the cost of cell phone service for a month is represented by the expression $0.40s + 12.95$**.** Students can analyze how the coefficient of 0.40 represents the cost of one minute (40¢), while the constant of 12.95 represents a fixed, monthly fee, and s stands for the number of cell phone minutes used in the month. Similar real-world examples, such as tax rates, can also be used to explore the meaning of expressions.

Factoring by grouping is another example of how students might analyze the structure of an expression**.** To factor $3x(x – 5) + 2(x – 5)$, students should recognize that the "$x – 5$" is common to both expressions being added, so it simplifies to $(3x + 2)(x – 5)$**.** Students should become comfortable with rewriting expressions in a variety of ways until a structure emerges.

Have students create their own expressions that meet specific criteria (e.g., number of terms factorable, difference of two squares, etc.) and verbalize how they can be written and rewritten in different forms. Additionally, pair/group students to share their expressions and rewrite one another's expressions.

Resources/Tools

Hands-on materials, such as algebra tiles, can be used to establish a visual understanding of algebraic expressions and the meaning of terms, factors and coefficients.
From the National Library of Virtual Manipulatives - Algebra Tiles – Visualize multiplying and factoring algebraic expressions using tiles.

Misconceptions

Students may believe that the use of algebraic expressions is merely the abstract manipulation of symbols. Use of real-world context examples to demonstrate the meaning of the parts of algebraic expressions is needed to counter this misconception.

Students may also believe that an expression cannot be factored because it does not fit into a form they recognize. They need help with reorganizing the terms until structures become evident.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Specific strategies for mathematics may include:

Technology may be useful to help a student recognize that two different expressions represent the same relationship. For example, since $(x – y)(x + y)$ can be rewritten as $x^2 – y^2$, they can put both expressions into a graphing calculator (or spreadsheet) and have it generate two tables (or two columns of one table), displaying the same output values for each expression.

Connections:

An introduction to the use of variable expressions and their meaning, as well as the use of variables and expressions in real-life situations is included in the Expressions and Equations Domain of Grade 7.

High School Conceptual Category: Algebra

Domain	Seeing the structure of expressions
Cluster	***Write expressions in equivalent forms to solve problems***
Standards	3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression a. Factor a quadratic expression to reveal the zeros of the function it defines b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be written as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate* if *the annual rate is 15%* 4. Derive the formula for the sum of a finite geometric series (when the common ratio is not 1), and use the formula to solve problems. For example, calculate mortgage payments.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

This cluster focuses on linking expressions and functions, i.e., creating connections between multiple representations of functional relations – the dependence between a quadratic expression and a graph of the quadratic function it defines, and the dependence between different symbolic representations of exponential functions. Teachers need to foster the idea that changing the forms of expressions, such as factoring or completing the square, or transforming expressions from one exponential form to another, are not independent algorithms that are learned for the sake of symbol manipulations. They are processes that are guided by goals (e.g., investigating properties of families of functions and solving contextual problems).

Factoring methods that are typically introduced in elementary algebra and the method of completing the square reveals attributes of the graphs of quadratic functions, represented by quadratic equations.

- The solutions of quadratic equations solved by factoring are the x – intercepts of the parabola or zeros of quadratic functions.
- A pair of coordinates (h, k) from the general form $f(x) = a(x - h)^2 + k$ represents the vertex of the parabola, where h represents a horizontal shift and k represents a vertical shift of the parabola $y = x^2$ from its original position at the origin.
- A vertex (h, k) is the minimum point of the graph of the quadratic function if a › 0 and is the maximum point of the graph of the quadratic function if a ‹ 0. Understanding an algorithm of completing the square provides a solid foundation for deriving a quadratic formula.

Translating among different forms of expressions, equations and graphs helps students to understand some key connections among arithmetic, algebra and geometry. The reverse thinking technique (a process that allows working backwards from the answer to the starting point) can be very effective. Have students derive information about a function's equation, represented in standard, factored or general form, by investigating its graph.

Offer multiple real-world examples of exponential functions. For instance, to illustrate an exponential decay, students need to recognize that in the equation for an automobile cost $C(t) = 20{,}000(0.75)^t$, the base is 0.75 and between 0 and 1 and the value of $20,000 represents the initial cost of an automobile that depreciates 25% per year over the course of t years.

Similarly, to illustrate exponential growth, in the equation for the value of an investment over time $A(t) = 10{,}000(1.03)^t$, where the base is 1.03 and is greater than 1; and the $10,000 represents the value of an investment when increasing in

value by 3% per year for *x* years.

A problem such as, "An amount of $100 was deposited in a savings account on January 1st each of the years 2010, 2011, 2012, and so on to 2019, with annual yield of 7%. What will be the balance in the savings account on January 1, 2020?" illustrates the use of a formula for a geometric series $\boldsymbol{S_n = \frac{g(1-r^n)}{(1-r)}}$ when S_n represents the value of the geometric series with the first term g, constant ration $r \neq 1$, and n terms.

Before using the formula, it might be reasonable to demonstrate the way the formula is derived,

$S_n = g + gr + gr^2 + gr^3 + \ldots gr^{n-1}$.

Multiply by r $\quad rS_n = gr + gr^2 + \ldots + gr^{n-1} + gr^n$

Subtract $\quad S - rS = g - gr^n$

Factor $\quad S(1 - r) = g(1 - r^n)$

Divide by $(1 - r)$ $\quad \boldsymbol{S_n = \frac{g(1-r^n)}{(1-r)}}$

The amount of the investment for January 1, 2020 can be found using: $100(1.07)^{10} + 100(1.07)^9 + \ldots + 100(1.07)$. If the first term of this geometric series is $g = 100(1.07)$, the ratio is 1.07 and the number of terms $n = 10$, the formula for the value of geometric series is:

$$\boldsymbol{S_{10} = \frac{g(1-r^{10})}{(1-r)} = \frac{100(1.07)(1.07^{10}-1)}{(1.07-1)} = \frac{107(1.07^{10}-1)}{0.07}}$$

$S_{10} \approx \$1478.36$

Instructional Resources/Tools

Graphing utilities to explore the effects of parameter changes on a graph
Tables, graphs and equations of real-world applications that apply quadratic and exponential functions
Computer algebra systems

From the National Library of Virtual Manipulatives - Grapher – A tool for graphing and exploring functions.

http://www.learner.org/workshops/algebra/workshop5/lessonplan1.html
This website contains a lesson and a workshop that showcases ways that teachers can help students explore mathematical properties studied in algebra. The activities use a variety of techniques to help students understand concepts of factoring quadratic trinomials.

From the National Council of Teachers of Mathematics, Illuminations - Difference of Squares - This activity uses a series of related arithmetic experiences to prompt students to explore arithmetic statements leading to a result that is the factoring pattern for the difference of two squares. A geometric interpretation of the familiar formula is also included.

Common Misconceptions

Some students may believe that factoring and completing the square are isolated techniques within a unit of quadratic equations. Teachers should help students to see the value of these skills in the context of solving higher degree equations and examining different families of functions.

Students may think that the minimum (the vertex) of the graph of $y = (x + 5)^2$ is shifted to the right of the minimum (the vertex) of the graph $y = x^2$ due to the addition sign. Students should explore examples both analytically and graphically to overcome this misconception.

Some students may believe that the minimum of the graph of a quadratic function always occur at the *y*-intercept.
Some students cannot distinguish between arithmetic and geometric sequences, or between sequences and series. To avoid this confusion, students need to experience both types of sequences and series.

Diverse Learners

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Specific strategies for mathematics may include:
Before using graphing technology to explore the effect of changing constants in quadratic equations, allow students to first make tables by hand.
Highlight and compare diverse algorithms to discuss different approaches to solving the same problem.

Connections:

In Grade 8, students compare tables, graphs, expressions and equations of linear relationships. In high school, examination of expressions and equations is intended to include quadratic and exponential, as well as not linear sequences and series. Students will recognize how the form of the expression or equation can provide information about its graph (vertex, minimum, maximum, etc).

Domain	**Arithmetic with Polynomials and Rational Expressions**
Cluster	**Perform arithmetic operations on polynomials**
Standards	1. Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

The primary strategy for this cluster is to make connections between arithmetic of integers and arithmetic of polynomials. In order to understand this standard, students need to work toward both understanding and fluency with polynomial arithmetic. Furthermore, to talk about their work, students will need to use correct vocabulary, such as integer, monomial, polynomial, factor, and term.

In arithmetic of polynomials, a central idea is the distributive property, because it is fundamental not only in polynomial multiplication but also in polynomial addition and subtraction. With the distributive property, there is little need to emphasize misleading mnemonics, such as FOIL, which is relevant only when multiplying two binomials, and the procedural reminder to “collect like terms” as a consequence of the distributive property. For example, when adding the polynomials 3x and 2x, the result can be explained with the distributive property as follows: 3x + 2x = (3 + 2)x = 5x.

An important connection between the arithmetic of integers and the arithmetic of polynomials can be seen by considering whole numbers in base ten place value to be polynomials in the base *b* = 10. For two-digit whole numbers and linear binomials, this connection can be illustrated with area models and algebra tiles. But the connections between methods of multiplication can be generalized further. For example, compare the product 213 x 47 with the product $(2b^2 + 1b + 3)(4b + 7)$:

$$\begin{array}{r} 2b^2 + 1b + 3 \\ \times \quad 4b + 7 \\ \hline 14b^2 + 7b + 21 \\ 8b^3 + 4b^2 + 12b \\ \hline 8b^3 + 18b^2 + 19b + 21 \end{array} \qquad \begin{array}{r} 200 + 10 + 3 \\ \times \quad 40 + 7 \\ \hline 1400 + 70 + 21 \\ 8000 + 400 + 120 \\ \hline 8000 + 1800 + 190 + 21 \end{array} \qquad \begin{array}{r} 213 \\ \times \quad 47 \\ \hline 1491 \\ 8520 \\ \hline 10011 \end{array}$$

Note how the distributive property is in play in each of these examples: In the left-most computation, each term in the factor $(4b + 7)$ must be multiplied by each term in the other factor, $(2b^2 + 1b + 3)$, just like the value of each digit in 47 must be multiplied by the value of each digit in 213, as in the middle computation, which is similar to “partial products methods” that some students may have used for multiplication in the elementary grades. The common algorithm on the right is merely an abbreviation of the partial products method.

The new idea in this standard is called *closure*: A set is *closed* under an operation if when any two elements are combined with that operation, the result is always another element of the same set. In order to understand that polynomials are closed under addition, subtraction and multiplication, students can compare these ideas with the analogous claims for integers: The sum, difference or product of any two integers is an integer, but the quotient of two integers is not always an integer.

Now for polynomials, students need to reason that the sum (difference or product) of any two polynomials is indeed a polynomial. At first, restrict attention to polynomials with integer coefficients. Later, students should consider polynomials with rational or real coefficients and reason that such polynomials are closed under these operations.

For contrast, students need to reason that polynomials are not closed under the operation of division: The quotient of two polynomials is not always a polynomial. For example $(x^2 + 1) \div x$ is not a polynomial. Of course, the quotient of two polynomials is sometimes a polynomial. For example, $(x^2 - 9) \div (x - 3) = x + 3$.

Instructional Resources/Tools

Graphing calculators
Graphing software, including dynamic geometry software
Computer Algebra Systems
Algebra tiles
Area models

Common Misconceptions

Some students will apply the distributive property inappropriately. Emphasize that it is the *distributive property of multiplication over addition*. For example, the distributive property can be used to rewrite $2(x + y)$ as $2x + 2y$, because in this product the second factor is a sum (i.e., involving addition). But in the product $2(xy)$, the second factor, (xy), is itself a product, not a sum.

Some students will still struggle with the arithmetic of negative numbers. Consider the expression $(-3) \cdot \big(2 + (-2)\big)$. On the one hand, $(-3) \cdot \big(2 + (-2)\big) = (-3) \cdot (0) = 0$. But using the distributive property, $(-3) \cdot \big(2 + (-2)\big) = (-3) \cdot (2) + (-3) \cdot (-2)$. Because the first calculation gave 0, the two terms on the right in the second calculation must be opposite in sign. Thus, if we agree that $(-3) \cdot (2) = -6$, then it must follow that $(-3) \cdot (-2) = 6$.

Diverse Learners

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Connections:

To further explore connections, students might look for commonalities between factoring polynomials and factoring integers. For example, some polynomials are factorable, just as some integers are factorable, and some polynomials are prime or not factorable, just as some integers are prime numbers.

Domain	Arithmetic with Polynomials and Rational Expressions
Cluster	***Understand the relationship between zeros and factors of polynomials***
Standards	2. Know and apply the Remainder Theorem: For a polynomial *p*(*x*) and a number *a*, the remainder on division by *x* – *a* is *p*(*a*), so *p*(*a*) = 0 if and only if (*x* – *a*) is a factor of *p*(*x*). 3. Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

As discussed for the previous cluster (**Perform arithmetic operations on polynomials**), polynomials can often be factored. Even though polynomials (i.e., polynomial expressions) can be explored as mathematical objects without consideration of functions, in school mathematics, polynomials are usually taken to define functions. Some equations may include polynomials on one or both sides. The importance here is in distinction between equations that have solutions, and functions that have zeros. Thus, polynomial functions have zeros. This cluster is about the relationship between the factors of a polynomial, the zeros of the function defined by the polynomial, and the graph of that function. The zeros of a polynomial function are the *x*-intercepts of the graph of the function.

Through some experience with long division of polynomials by $(x - a)$, students get a sense that the quotient is always a polynomial of a polynomial that is one degree less than the degree of the original polynomial, and that the remainder is always a constant. In other words, $p(x) = q(x)(x - a) + r$. Using this equation, students reason that $p(a) = r$. Thus, if $p(a) = 0$, then the remainder $r = 0$, the polynomial $p(x)$ is divisible by $(x - a)$ and $(x - a)$ is a factor of $p(x)$. Conversely, if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$.

Whereas, the first standard specifically targets the relationship between factors and zeros of polynomials, the second standard requires more general exploration of polynomial functions: graphically, numerically, verbally and symbolically.

Through experience graphing polynomial functions in factored form, students can interpret the Remainder Theorem in the graph of the polynomial function. Specifically, when $x - a$ is a factor of a polynomial $p(x)$, then $p(a) = 0$, and therefore $x = a$ is an *x*-intercept of the graph $y = p(x)$. Conversely, when students notice an *x*-intercept near $x = b$ in the graph of a polynomial function $p(x)$, then the function has a zero near $x = b$, and $p(b)$ is near zero. Zeros are located approximately when reasoning from a graph. Therefore, if $p(b)$ is not exactly zero, then $(x - b)$ is not a factor of $p(x)$.

Students can benefit from exploring the rational root theorem, which can be used to find all of the possible rational roots (i.e., zeros) of a polynomial with integer coefficients. When the goal is to identify all roots of a polynomial, including irrational or complex roots, it is useful to graph the polynomial function to determine the most likely candidates for the roots of the polynomial that are the x-intercepts of the graph.

When at least one rational root $x = r_1$ is identified, the original polynomial can be divided by $x - r_1$, so that additional roots can be sought in the quotient. Long division will suffice in simple cases. Synthetic division is an abbreviated method that is less prone to error in complicated cases, but Computer Algebra Systems may be helpful in such cases. Graphs are used to understand the end-behavior of n^{th} degree polynomial functions, to locate roots and to infer the existence of complex roots. By using technology to explore the graphs of many polynomial functions, and describing the shape, end behavior and number of zeros, students can begin to make the following informal observations:

- The graphs of polynomial functions are continuous.
- An n^{th} degree polynomial has at most *n* roots and at most *n* - 1 “changes of direction” (i.e., from increasing to decreasing or vice versa).

- An even-degree polynomial has the same end-behavior in both the positive and negative directions: both heading to positive infinity, or both heading to negative infinity, depending upon the sign of the leading coefficient.
- An odd-degree polynomial has opposite end-behavior in the positive versus the negative directions, depending upon the sign of the leading coefficient.
- An odd-degree polynomial function must have at least one real root.

Instructional Resources/Tools

Graphing calculators
Graphing software, including dynamic geometry software
Computer Algebra Systems

Common Misconceptions

Diverse Learners

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Connections:

Complex Numbers (N-CN.9): With complex numbers and when roots are counted according to their multiplicity, the Fundamental Theorem of Algebra states that an nth degree polynomial has exactly n roots. With this theorem, students can predict the number of real and non-real complex roots of a polynomial based on its graph.

Domain	Arithmetic with Polynomials and Rational Expressions
Cluster	***Use polynomial identities to solve problems***
Standards	4. Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 – y^2)^2 + (2xy)^2$ *can be used to generate Pythagorean triples.* 5. (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

In Grade 6, students began using the properties of operations to rewrite expressions in equivalent forms. When two expressions are equivalent, an equation relating the two is called an *identity* because it is true for all values of the variables. This cluster is an opportunity to highlight polynomial identities that are commonly used in solving problems. To learn these identities, students need experience using them to solve problems.

Students should develop familiarity with the following special products:

$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x - y)^2 = x^2 - 2xy + y^2$$
$$(x + y)(x - y) = x^2 - y^2$$
$$(x + a)(x + b) = x^2 + (a + b)x + ab$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 – y^3$$

Students should be able to prove any of these identities. Furthermore, they should develop sufficient fluency with the first four of these so that they can recognize expressions of the form on either side of these identities in order to replace that expression with an equivalent expression in the form of the other side of the identity.

- With identities such as these, students can discover and explain facts about the number system. For example, in the multiplication table, the perfect squares appear on the diagonal. Diagonally, next to the perfect squares are "near squares," which are one less than the perfect square. Why?
- Why is the sum of consecutive odd numbers beginning with 1 always a perfect square?
- Numbers that can be expressed as the sum of the counting numbers from 1 to n are called triangular numbers. What do you notice about the sum of two consecutive triangular numbers? Explain why it works.
- The sum and difference of cubes are also reasonable for students to prove. The focus of this proof should be on generalizing the difference of cubes formula with an emphasis toward finite geometric series.

Some information below includes additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics and goes beyond the mathematics that all students should study in order to be college- and career-ready:

Ask students to use the vertical multiplication format (as describe in the first cluster) to write out term-by-term multiplication to generate $(x + y)^3$ from the expanded form of $(x + y)^2$. Then use that expanded result to expand $(x + y)^4$, use that result to expand $(x + y)^5$, and so on. Students should begin to see the arithmetic that generates the entries in Pascal's triangle.

Instructional Resources/Tools

Graphing calculators
Graphing software, including dynamic geometry software
Computer Algebra Systems

Common Misconceptions

Diverse Learners

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Connections:

When students "see structure in expressions," they are more likely to use these identities productively in solving problems.

Domain	Arithmetic with Polynomials and Rational Expressions
Cluster	*Rewrite rational expressions*
Standards	6. Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$, where $a(x)$, $b(x)$, $q(x)$, and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$, using inspection, long division, or, for the more complicated examples, a computer algebra system. 7. (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

This cluster is the logical extension of the earlier standards on polynomials and the connection to the integers. Now, the arithmetic of rational functions is governed by the same rules as the arithmetic of fractions, based first on division.

In particular, in order to write $\frac{a(x)}{b(x)}$ in the form $q(x) + \frac{r(x)}{b(x)}$, students need to work through the long division described for A-APR.2-3. This is merely writing the result of the division as a quotient and a remainder. For example, we can rewrite $\frac{75}{8}$ in the form $9 + \frac{3}{8}$. Note that the fraction $\frac{75}{8}$ is interpreted as the division $75 \div 8$, so that 75 is the dividend and 8 is the divisor. The result indicates that 9 is the quotient and 3 is the remainder. Note that for division of integers, we expect the remainder to be between 0 and the divisor, which in this case is 8. (If the remainder were greater than or equal to 8, we could subtract another 8, and increase the quotient by 1.)

In order to rewrite simple rational expressions in different forms, students need to understand that the rules governing the arithmetic of rational expressions are the same rules that govern the arithmetic of rational numbers (i.e., fractions). To add fractions, we use a common denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

as long as $b, d \neq 0$. Although in simple situations, a, b, c, and d would each be whole numbers, in fact they can be polynomials. So now suppose that $a = 2, b = (x - 1), c = x,$ and $d = (x + 1)$, then

$$\frac{2}{x-1} + \frac{x}{x+1} = \frac{2(x+1)}{(x-1)(x+1)} + \frac{(x-1)x}{(x-1)(x+1)} = \frac{2(x+1) + (x-1)x}{(x-1)(x+1)}$$

And then the numerator can be simplified further:

$$= \frac{2x + 2 + x^2 - x}{(x-1)(x+1)} = \frac{x^2 + x + 2}{(x-1)(x+1)}$$

In order to meet A-APR.6, students will need some experiences with the arithmetic of simple rational expressions. For most students, the above example helps illustrating the similarity of the form of the arithmetic used with rational expressions and the form of the arithmetic used with rational numbers. As indicated by the (+) symbol, some (but not all) students will need to develop fluency with these skills.

Instructional Resources/Tools

Graphing calculators
Graphing software, including dynamic geometry software

Computer Algebra Systems

Common Misconceptions

Students with only procedural understanding of fractions are likely to cancel terms (rather than factors of) in the numerator and denominator of a fraction. Emphasize the structure of the rational expression: that the *whole numerator* is divided by the *whole denominator*. In fact, the word "cancel" likely promotes this misconception. It would be more accurate to talk about dividing the numerator and denominator by a common factor.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Connections:

Seeing Structure in Expressions. The arithmetic of rational expressions is fundamentally about seeing the same structure in rational expressions as the arithmetic of rational numbers (i.e., fractions).

High School Conceptual Category: Algebra

Domain	Creating Equations
Cluster	***Create equations that describe numbers or relationships***
Standards	1. Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.* 2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. 3. Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods*. 4. Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

As the framework for the assessments, this section will be developed by the CCSS assessment consortia (SBAC and PARCC). Ohio is currently participating in both consortia and has input into the development of the frameworks. This information will be included as it is developed.

Instructional Strategies and Resources

Instructional Strategies

Provide examples of real-world problems that can be modeled by writing an equation or inequality. Begin with simple equations and inequalities and build up to more complex equations in two or more variables that may involve quadratic, exponential or rational functions.

Discuss the importance of using appropriate labels and scales on the axes when representing functions with graphs.

Examine real-world graphs in terms of constraints that are necessary to balance a mathematical model with the real-world context. For example, a student writing an equation to model the maximum area when the perimeter of a rectangle is 12 inches should recognize that $y = x(6 - x)$ only makes sense when $0 < x < 6$. This restriction on the domain is necessary because the side of a rectangle under these conditions cannot be less than or equal to 0, but must be less than 6. Students can discuss the difference between the parabola that models the problem and the portion of the parabola that applies to the context.

Explore examples illustrating when it is useful to rewrite a formula by solving for one of the variables in the formula. For example, the formula for the area of a trapezoid ($A = \frac{1}{2}h(b_1 + b_2)$) can be solved for *h* if the area and lengths of the bases are known but the height needs to be calculated. This strategy of selecting a different representation has many applications in science and business when using formulas .

Provide examples of real-world problems that can be solved by writing an equation, and have students explore the graphs of the equations on a graphing calculator to determine which parts of the graph are relevant to the problem context.

Use a graphing calculator to demonstrate how dramatically the shape of a curve can change when the scale of the graph is altered for one or both variables.

Give students formulas, such as area and volume (or from science or business), and have students solve the equations for each of the different variables in the formula.

Instructional Resources/Tools

Graphing calculators
Computer software that generate graphs of functions
Examples of real-world situations that lend themselves to writing equations that model the contexts.

Common Misconceptions

Students may believe that equations of linear, quadratic and other functions are abstract and exist only "in a math book," without seeing the usefulness of these functions as modeling real-world phenomena.

Additionally, they believe that the labels and scales on a graph are not important and can be assumed by a reader, and that it is always necessary to use the entire graph of a function when solving a problem that uses that function as its model.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Connections:

Working with expressions and equations, including formulas, is an integral part of the curriculum in Grades 7 and 8. In high school, students explore in more depth the use of equations and inequalities to model real-world problems, including restricting domains and ranges to fit the problem's context, as well as rewriting formulas for a variable of interest.

Domain	Reasoning with Equations and Inequalities
Cluster	***Understand solving equations as a process of reasoning and explain the reasoning***
Standards	1. Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method. 2. Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

Content Elaborations (in development)

This section will provide additional clarification and examples to aid in the understanding of the standards. To support shared interpretations across states, content elaborations are being developed through multistate partnerships organized by CCSSO and other national organizations. This information will be included as it is developed.

Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Challenge students to justify each step of solving an equation. Transforming $2x - 5 = 7$ to $2x = 12$ is possible because $5 = 5$, so adding the same quantity to both sides of an equation makes the resulting equation true as well. Each step of solving an equation can be defended, much like providing evidence for steps of a geometric proof.

Provide examples for how the same equation might be solved in a variety of ways as long as equivalent quantities are added or subtracted to both sides of the equation, the order of steps taken will not matter.

$3n + 2 = n - 10$
$-\quad 2 = \quad -2$
$3n \quad = n - 12$
$-n \quad = -n$
$2n \quad = -12$
$n = -6$

OR

$3n + 2 = n - 10$
$+\quad 10 = +10$
$3n + 12 = n$
$-3n \quad = -3n$
$12 = -2n$
$n = -6$

OR

$3n + 2 = n - 10$
$-n \quad = -n$
$2n + 2 = -10$
$\quad - 2 = -2$
$2n = -12$
$n = -6$

Connect the idea of adding two equations together as a means of justifying steps of solving a simple equation to the process of solving a system of equations. A system consisting of two linear functions such as $2x + 3y = 8$ and $x - 3y = 1$ can be solved by adding the equations together, and can be justified by exactly the same reason that solving the equation $2x - 4 = 5$ can begin by adding the equation $4 = 4$.

Investigate the solutions to equations such as $\mathbf{3} = \boldsymbol{x} + \sqrt{\mathbf{2}\boldsymbol{x} - \mathbf{3}}$. By graphing the two functions, $y = 3$ and $\boldsymbol{y} = \boldsymbol{x} + \sqrt{\mathbf{2}\boldsymbol{x} - \mathbf{3}}$, students can visualize that graphs of the functions only intersect at one point. However, subtracting $x = x$ from the original equation yields $\mathbf{3} - \boldsymbol{x} = \sqrt{\mathbf{2}\boldsymbol{x} - \mathbf{3}}$ which when both sides are squared produces a quadratic equation that has two roots $x = 2$ and $x = 6$. Students should recognize that there is only one solution ($x = 2$) and that $x = 6$ is generated when a quadratic equation results from squaring both sides; $x = 6$ is extraneous to the original equation. Some rational equations, such as $\frac{\boldsymbol{x}}{(\boldsymbol{x}-\mathbf{2})} = \frac{\mathbf{2}}{(\boldsymbol{x}-\mathbf{2})} + \frac{\mathbf{5}}{\boldsymbol{x}}$. result in extraneous solutions as well.

Begin with simple, one-step equations and require students to write out a justification for each step used to solve the equation.

Ensure that students are proficient with solving simple rational and radical equations that have no extraneous solutions before moving on to equations that result in quadratics and possible solutions that need to be eliminated.

Provide visual examples of radical and rational equations with technology so that students can see the solution as the intersection of two functions and further understand how extraneous solutions do not fit the model.

It is very important that students are able to reason how and why extraneous solutions arise.

Instructional Resources/Tools

Graphing calculators
Computer software that generates graphs for visually examining solutions to equations, particularly rational and radical
Examples of radical equations that do and do not result in the generation of extraneous solutions should be prepared for exploration

Common Misconceptions

Students may believe that solving an equation such as $3x+1=7$ involves "only removing the 1," failing to realize that the equation 1 = 1 is being subtracted to produce the next step.

Additionally, students may believe that all solutions to radical and rational equations are viable, without recognizing that there are times when extraneous solutions are generated and have to be eliminated.

Diverse Learners

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Connections:

Solving linear equations in one variable and analyzing pairs of simultaneous linear equations are part of the Grade 8 curriculum. In high school, students extend these ideas into radical and rational equations, including justification of steps taken to solve equations and recognition of extraneous solutions when they occur.

High School Conceptual Category: Algebra

Domain	Reasoning with Equations and Inequalities
Cluster	***Solve equations and inequalities in one variable***
Standards	3. Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. 4. Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. b. Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them in $a \pm bi$ for real numbers a and b.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

There are two major reasons for discussing the topic of inequalities along with equations: First, there are analogies between solving equations and inequalities that help students understand them both**.** Second, the applications that lead to equations almost always lead in the same way to inequalities.

In grades 6-8, students solve and graph linear equations and inequalities. Graphing experience with inequalities is limited to graphing on a number line diagram. Despite this work, some students will still need more practice to be proficient. It may be beneficial to remind students of the most common solving techniques, such as converting fractions from one form to another, removing parentheses in the sentences, or multiplying both sides of an equation or inequality by the common denominator of the fractions. Students must be aware of what it means to check an inequality's solution. The substitution of the end points of the solution set in the original inequality should give equality regardless of the presence or the absence of an equal sign in the original sentence**.** The substitution of any value from the rest of the solution set should give a correct inequality.

Careful selection of examples and exercises is needed to provide students with meaningful review and to introduce other important concepts, such as the use of properties and applications of solving linear equations and inequalities. Stress the idea that the application of properties is also appropriate when working with equations or inequalities that include more than one variable, fractions and decimals. Regardless of the type of numbers or variables in the equation or inequality, students have to examine the validity of each step in the solution process.

Solving equations for the specified letter with coefficients represented by letters (e.g., $A = \frac{1}{2}h(B + b)$, when solving for b) is similar to solving an equation with one variable. Provide students with an opportunity to abstract from particular numbers and apply the same kind of manipulations to formulas as they did to equations. One of the purposes of doing abstraction is to learn how to evaluate the formulas in easier ways and use the techniques across mathematics and science.

Draw students' attention to equations containing variables with subscripts. The same variables with different subscripts (e.g., x_1 and x_2) should be viewed as different variables that cannot be combined as like terms. A variable with a variable subscript, such as a_n, must be treated as a single variable – the nth term, where variables a and n have different meaning.

Completing the square is usually introduced for several reasons" to find the vertex of a parabola whose equation has been expanded; to look at the parabola through the lenses of translations of a "parent" parabola $y = x^2$; and to derive a quadratic formula. Completing the square is a very useful tool that will be used repeatedly by students in many areas of

mathematics. Teachers should carefully balance traditional paper-pencil skills of manipulating quadratic expressions and solving quadratic equations along with an analysis of the relationship between parameters of quadratic equations and properties of their graphs.

Start by inspecting equations such as $x^2 = 9$ that has two solutions, 3 and -3. Next, progress to equations such as $(x - 7)^2 = 9$ by substituting $x - 7$ for x and solving them either by "inspection" or by taking the square root on each side:

$x - 7 = 3$ and $x - 7 = -3$
$x = 10$ $x = 4$

Graph both pairs of solutions (-3 and 3, 4 and 10) on the number line and notice that 4 and 10 are 7 units to the right of – 3 and 3. So, the substitution of $x - 7$ for x moved the solutions 7 units to the right. Next, graph the function $y = (x - 7)^2 - 9$, pointing out that the x-intercepts are 4 and 10, and emphasizing that the graph is the translation of 7 units to the right and 9 units down from the position of the graph of the parent function $y = x^2$ that passes through the origin (0, 0). Generate more equations of the form $y = a(x - h)^2 + k$ and compare their graphs using a graphing technology.

Highlight and compare different approaches to solving the same problem. Use technology to recognize that two different expressions or equations may represent the same relationship. For example, since $x^2 - 10x + 25 = 0$ can be rewritten as $(x - 5)(x - 5) = 0$ or $(x - 5)^2 = 0$ or $x^2 = 25$, these are all representations of the same equation that has a double solution x = 5. Support it by putting all expressions into graphing calculator. Compare their graphs and generate their tables displaying the same output values for each expression.

Guide students in transforming a quadratic equation in standard form, $0 = ax^2 + bx + c$, to the vertex form $0 = a(x - h)^2 + k$ by separating your examples into groups with $a = 1$ and $a \neq 1$ and have students guess the number that needs to be added to the binomials of the type $x^2 + 6x$, $x^2 - 2x$, $x^2 + 9x$, $x^2 - \frac{2}{3}x$ to form complete square of the binomial $(x - m)^2$. Then generalize the process by showing the expression $(b/2)^2$ that has to be added to the binomial $x^2 + bx$. Completing the square for an expression whose x^2 coefficient is not 1 can be complicated for some students. Present multiple examples of the type $0 = 2x^2 - 5x - 9$ to emphasize the logic behind every step, keeping in mind that the same process will be used to complete the square in the proof of the quadratic formula.

Discourage students from giving a preference to a particular method of solving quadratic equations. Students need experience in analyzing a given problem to choose an appropriate solution method before their computations become burdensome. Point out that the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is a universal tool that can solve any quadratic equation; however, it is not reasonable to use the Quadratic Formula when the quadratic equation is missing either a middle term, bx, or a constant term, c. When it is missing a constant term, (e.g., $3x^2 - 10x = 0$) a factoring method becomes more efficient. If a middle term is missing (e.g., $2x^2 - 15 = 0$), a square root method is the most appropriate. Stress the benefit of memorizing the Quadratic Formula and flexibility with a factoring strategy. Introduce the concept of discriminants and their relationship to the number and nature of the roots of quadratic equation.

, Offer students examples of a quadratic equation, such as $x^2 + 9 = 0$. Since the graph of the quadratic function $y = x^2 + 9$ is situated above the x –axis and opens up, the graph does not have x–intercepts and therefore, the quadratic equation does not have real solutions. At this stage introduce students to non-real solutions, such as $x = \pm\sqrt{-9}$ or $x = \pm 3\sqrt{-1}$ and a new number type-imaginary unit i that equals $\sqrt{-1}$. Using i in place of $\sqrt{-1}$ is a way to present the two solutions of a quadratic equation in the complex numbers form $a \pm bi$, if a and b are real numbers and $b \neq 0$. Have students observe that if a quadratic equation has complex solutions, the solutions always appear in conjugate pairs, in the form $a + bi$ and $a - bi$. Particularly, for the equation $x^2 = -9$, a conjugate pair of solutions are $0 + 3i$ and $0 - 3i$. Project the same logic in the examples of any quadratic equations $0 = ax^2 + bx + c$ that have negative discriminants. The solutions are a pair of conjugate complex numbers $\frac{-b \pm i\sqrt{D}}{2a}$, if $D = b^2 - 4ac$ is negative.

Instructional Resources/Tools

Graphing utilities to explore the effects of changes in parameters of equations on their graphs
Tables, graphs and equations of real-world applications that apply quadratic and exponential functions

http://ohiorc.org/for/math/
http://illuminations.nctm.org/

Common Misconceptions

Some students may believe that for equations containing fractions only on one side, it requires "clearing fractions" (the

use of multiplication) only on that side of the equation. To address this misconception, start by demonstrating the solution methods for equations similar to $\frac{1}{4}x + \frac{1}{5}x + \frac{1}{6}x + 46 = x$ and stress that the Multiplication Property of Equality is applied to both sides, which are multiplied by 60.

Students may confuse the rule of changing a sign of an inequality when multiplying or dividing by a negative number with changing the sign of an inequality when one or two sides of the inequality become negative (for ex., $3x > -15$ or $x < -5$).

Some students may believe that subscripts can be combined as $b_1 + b_2 = b_3$ and the sum of different variables *d* and D is 2D (*d* +D = 2D).

Some students may think that rewriting equations into various forms (taking square roots, completing the square, using quadratic formula and factoring) are isolated techniques within a unit of quadratic equations. Teachers should help students see the value of these skills in the context of solving higher degree equations and examining different families of functions.

Diverse Learners

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Connections

In grades 6-8, students learned how to approach linear equations in which justification of procedures was the basis for proofs. In high school, based on experience gained in solving quadratic equations, students will understand the need for a variety of methods when solving other types of equations, including conics (i.e., ellipses, parabolas, hyperbolas).

High School Conceptual Category: Algebra

Domain	Reasoning with Equations and Inequalities
Cluster	***Solve systems of equations***
Standards	5. Prove that a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions. 6. Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables. 7. Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line y = -3x and the circle $x^2 + y^2 = 3$. 8. (+) Represent a system of linear equations as a single matrix equation in a vector variable. 9. (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension 3x3 or greater)

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

The focus of this standard is to provide mathematics justification for the addition (elimination) and substitution methods of solving systems of equations that transform a given system of two equations into a simpler equivalent system that has the same solutions as the original.

The Addition and Multiplication Properties of Equality allow finding solutions to certain systems of equations. In general, any linear combination, $m(Ax + By) + n(Cx + Dy) = mE + nF$, of two linear equations

$Ax + By = E$ and

$Cx + Dy = F$

intersecting in a single point contains that point. The multipliers m and n can be chosen so that the resulting combination has only an *x*-term or only a *y*-term in it. That is, the combination will be a horizontal or vertical line containing the point of intersection.

In the proof of a system of two equations in two variables, where one equation is replaced by the sum of that equation and a multiple of the other equation, produces a system that has the same solutions, let point (x_1, y_1) be a solution of both equations:

$Ax_1 + By_1 = E$ (true)

$Cx_1 + Dy_1 = F$ (true)

Replace the equation $Ax + By = E$ with $Ax + By + k(Cx + Dy)$ on its left side and with $E + kF$ on its right side.

The new equation is $Ax + By + k(Cx + Dy) = E + kF$.

Show that the ordered pair of numbers (x_1, y_1) is a solution of this equation by replacing (x_1, y_1) in the left side of this equation and verifying that the right side really equals $E + kF$:

$$Ax_1 + By_1 + k(Cx_1 + Dy_1) = E + kF \text{ (true)}$$

Systems of equations are classified into two groups, consistent or inconsistent, depending on whether or not solutions exist. The solution set of a system of equations is the intersection of the solution sets for the individual equations. Stress the benefit of making the appropriate selection of a method for solving systems (graphing vs. addition vs. substitution). This depends on the type of equations and combination of coefficients for corresponding variables, without giving a preference to either method.

The graphing method can be the first step in solving systems of equations. .A set of points representing solutions of each equation is found by graphing these equations. Even though the graphing method is limited in finding exact solutions and often yields approximate values, the use of it helps to discover whether solutions exist and, if so, how many are there

Prior to solving systems of equations graphically, students should revisit "families of functions" to review techniques for graphing different classes of functions. Alert students to the fact that if one equation in the system can be obtained by multiplying both sides of another equation by a nonzero constant, the system is called consistent, the equations in the system are called dependent and the system has an infinite number of solutions that produces coinciding graphs. Provide students opportunities to practice linear vs. non-linear systems; consistent vs. inconsistent systems; pure computational vs. real-world contextual problems (e.g., chemistry and physics applications encountered in science classes). A rich variety of examples can lead to discussions of the relationships between coefficients and consistency that can be extended to graphing and later to determinants and matrices.

The next step is to turn to algebraic methods, elimination or substitution, to allow students to find exact solutions. For any method, stress the importance of having a well organized format for writing solutions.

Information below contains additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics, and goes beyond the mathematics that all students should study in order to be the college- and career-ready:

Some students may wonder why they learn to use matrices to solve systems of linear equations if solutions can be found using previously learned methods. Methods associated with matrices are used by computers and generalize solving systems with many equations and many variables. Provide examples of real-world situations that can be solved by writing systems of equation or making matrices, and have students explore the graphs of the equations on the calculator to determine the relevancy of the graphs and units to the problem context.

Allow time for students to investigate connections between matrices and transformations, both algebraically and graphically, and verify conjectures about transformations.

A system of linear equations can be considered as a single matrix equation A•X = B, where A is a matrix of coefficients, X is a matrix of variables to be found, and B is the matrix of constants.

For example, a system of equations $2x - y = -9$
$3x - 8y = -7$

can be rewritten as

$$\begin{bmatrix} 2 & -1 \\ 3 & -8 \end{bmatrix} \bullet \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -9 \\ -7 \end{bmatrix}.$$

(Note, the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ is a column vector, introduced in the operations of vectors and matrices in the Number and Quantity Conceptual Category.)

Connect the strategy of solving linear equations by applying inverse operations to both sides of equations to solve linear systems by multiplying a matrix equation by the inverse of the coefficient matrix. In the 2x2 case, advise students to use the formula for finding the inverse of the coefficient matrix, if it exists. For a 3x3 case, either provide students with an inverse matrix, or show them how to find it by using a graphing tool. Reemphasize that the inverse matrix must always be written to the left of the constant matrix in order to multiply the matrices. Otherwise, the multiplication cannot be done. Once again, it is important to show a connection between the matrices and real numbers. Just as the two real numbers 1 and -1 are their own multiplicative inverses, the identity matrix is its own inverse.

Instructional Resources/Tools

Graph paper
Graphing calculators
Computer graphing tools to operate with matrices and to find determinants of higher order matrices.
Dynamic geometry software

From the National Council of Teachers of Mathematics, Illuminations: Supply and Demand - This activity focuses on having students create and solve a system of linear equations in a real-world setting. By solving a system of two equations in two unknowns, students will find the equilibrium point for supply and demand.

http://www.nsa.gov/academia/_files/collected_learning/high_school/algebra/making_connections.pdf
Students use graphing calculators to solve systems of linear equations in two ways. They first solve the systems by graphing the equations and finding the point of intersection. Next, they will solve systems of equations by writing related matrices and finding the solution by using inverse matrices.

From the National Council of Teachers of Mathematics, Illuminations: Movement with Functions - In this lesson, students use remote-controlled cars to create a system of equations.

Students use matrices and technology to solve the Meadows or Malls problem, a linear programming problem with six variables.

Common Misconceptions

Most mistakes that students make are careless rather than conceptual. Teachers should encourage students to learn a certain format for solving systems of equations and check the answers by substituting into all equations in the system. Some students believe that matrices are independent of other areas of mathematics. Teachers should establish the conceptual linkage between matrices, systems of equations and vectors.

Diverse Learners

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Connections

Students use their experience in solving and analyzing systems of two linear equations as a foundation for solving and analyzing linear systems with more than two linear equations and systems with non-linear equations.

Systems of linear equations can be studied in terms of matrices and matrices can be studied in terms of transformations.

Domain	Reasoning with Equations and Inequalities
Cluster	***Represent and solve equations and inequalities graphically***
Standards	10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line). 11. Explain why the *x*-coordinates of the points where the graphs of the equations *y = f(x)* and *y = g(x)* intersect are the solutions of the equation *f(x) = g(x)*; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where *f(x)* and/or *g(x)* are linear, polynomial, rational, absolute value, exponential, and logarithmic functions. 12. Graph the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Content Elaborations (in development)

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Expectations for Learning (in development)

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Instructional Strategies and Resources

Instructional Strategies

Beginning with simple, real-world examples, help students to recognize a graph as a set of solutions to an equation. For example, if the equation *y* = 6x + 5 represents the amount of money paid to a babysitter (i.e., $5 for gas to drive to the job and $6/hour to do the work), then every point on the line represents an amount of money paid, given the amount of time worked.

Explore visual ways to solve an equation such as 2*x* + 3 = x – 7 by graphing the functions *y = 2x* + 3 and *y = x* – 7. Students should recognize that the intersection point of the lines is at (-10, -17). They should be able to verbalize that the intersection point means that when *x* = -10 is substituted into both sides of the equation, each side simplifies to a value of -17. Therefore, -10 is the solution to the equation. This same approach can be used whether the functions in the original equation are linear, nonlinear or both.

Using technology, have students graph a function and use the trace function to move the cursor along the curve. Discuss the meaning of the ordered pairs that appear at the bottom of the calculator, emphasizing that every point on the curve represents a solution to the equation.
Begin with simple linear equations and how to solve them using the graphs and tables on a graphing calculator. Then, advance students to nonlinear situations so they can see that even complex equations that might involve quadratics, absolute value, or rational functions can be solved fairly easily using this same strategy. While a standard graphing calculator does not simply solve an equation for the user, it can be used as a tool to approximate solutions.

Use the table function on a graphing calculator to solve equations. For example, to solve the equation x^2 = x + 12, students can examine the equations $y = x^2$ *and* $y = x + 12$ and determine that they intersect when *x* = 4 and when *x* = -3 by examining the table to find where the *y*-values are the same.

Investigate real-world examples of two-dimensional inequalities. For example, students might explore what the graph would look like for money earned when a person earns *at least* $6 per hour. (The graph for a person earning *exactly* $6/hour would be a linear function, while the graph for a person earning at least $6/hour would be a half-plane including the line and all points above it.) Applications such as linear programming can help students to recognize how businesses use constraints to maximize profit while minimizing the use of resources. These situations often involve the use of systems of two variable inequalities.

Instructional Resources/Tools

Examples of real-world situations that involve linear functions and two-variable linear inequalities
Graphing calculators or computer software that generate graphs and tables for solving equations

Common Misconceptions

Students may believe that the graph of a function is simply a line or curve "connecting the dots," without recognizing that the graph represents all solutions to the equation.

Students may also believe that graphing linear and other functions is an isolated skill, not realizing that multiple graphs can be drawn to solve equations involving those functions.

Additionally, students may believe that two-variable inequalities have no application in the real world. Teachers can consider business related problems (e.g., linear programming applications) to engage students in discussions of how the inequalities are derived and how the feasible set includes all the points that satisfy the conditions stated in the inequalities.

Diverse Learners

Information and instructional strategies for gifted students, English Language Learners (ELL), and students with disabilities is available in the Introduction to Universal Design for Learning document located on the Revised Academic Content Standards and Model Curriculum Development Web page. Additional strategies and resources based on the Universal Design for Learning principles can be found at www.cast.org.

Connections:

Solving linear equations in one variable and analyzing pairs of simultaneous linear equations is part of the Grade 8 curriculum. These ideas are extended in high school, as students explore paper-and-pencil and graphical ways to solve equations, as well as how to graph two variable inequalities and solve systems of inequalities.